11. Grad, H. Note on n-dimensional Hermite polynomials. Comm. Pure Appl. Math. Vol. 2, №4, 1949.

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ON THE APPLICATION OF THE ENSKOG METHOD TO THE BOLTZMANN EQUATION

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Perusal of the Enskog-Chpman method appearing in its usual form in [1 and 2] or discussed in [3] shows clearly that the Enskog series for the distribution function satisfying the Boltzmann equation is asymptotic. This representation of the distribution function can also be deduced from the integral equation obtained by integrating the left and right-hand sides of the Boltzmann equation along the trajectory of the molecule (see e. g. [2]). Nevertheless, the asymptotic character of the Enskog expansion becomes particularly clear, if the integral form of the kinetic equation containing the probabilities of free paths of the molecules is used, and the Laplace's method applied in the asymptotic estimate of the integral.

We also see that the region of applicability of the resulting asymptotic representation is substantially curtailed (on the side of the high order of rarefaction) by discarding the corresponding exponential terms characterizing in particular the influence of the initial and boundary conditions.

1. We shall write the kinetic equation in the form

$$f(t, x_i, \xi_i) = f(t_0, x_{0i}, \xi_i) \exp\left(-\int_{t}^{t} N_\tau d\tau\right) + \tag{1.1}$$

$$+ \int_{t_0}^{t} f^+(\tau, x_i - \xi_i(t-\tau), \xi_i) N_\tau \exp\left(-\int_{\tau}^{t} N_q dq\right) d\tau$$

$$N(t, x_i, \xi_i) = \int f(t, x_i, \xi_{i1}) \sigma g d\xi_1, \qquad J^+ = \int f' f_1' g d\sigma d\xi_1$$

$$g = |\xi - \xi_1|, \qquad N_\tau = N(\tau, x_i - \xi_i(t-\tau), \xi_i), \qquad f^+ = J^+ / N$$

Here N denotes the collision frequency, J^+ is the integral of the reverse collisions, σ is the collision cross section and $t - t_0$ is the time of the collisionless passage of a molecule with velocity ξ_i from some point x_{0i} to the point $x_i = x_{0i} + \xi_i(t - \tau)$ under consideration.

Equations (1.1) show that the velocity distribution of the molecules at any instant t and at an arbitrary point x_i, is related to the values of the distribution function at any points of the region situated arbitrarily far from the point x_i, and at any instant of time preceding t. Nevertheless, the degree of mutual dependence of two points x_i and x_{0i} and of two instants t and t_0, decreases exponentially with increasing both the distance between these points and the time interval $t - t_0$, and the rate of decrease is directly related to the collision frequency. Our aim will be to find an explicit expression for the distribution function satisfying the kinetic equation, under the condition that the velocity distribution of the molecules at the point t, x_i is defined, basically, by the behavior of f in sufficiently near vicinity of this point.

We shall begin by analysing the asymptotic behavior of the right side of (1. 1) noting that its second term is, in fact, a Laplace integral [4]. To make the resemblance complete, one could introduce a large value parameter in its exponential index by incorporating the necessary scale factors. This, however, is not really necessary, since the validity of the asymptotic estimate depends only on the relative behavior of the integrand functions, and our concern is to ensure that the change in the value of f^+ is sufficiently weak and does not appreciably influence the behavior of the exponential term.

We shall consider the time t and the points x_i both sufficiently removed from the initial time and the boundary, respectively, i. e. we assume that the influence of the initial and boundary conditions on the state of gas at the instant t and at the point x_i, is vanishingly small. We shall call such points x_{0i} internal (not lying on the boundary) and $t_0 > 0$. The value of t_0 in (1. 1) which we shall choose for the considered t, x_i and ξ_i may, in general, be arbitrary (for the steady state problems we choose t_0 according to the given point x_{0i}).

When t_0 is close to t and ξ_i is fixed, then the exponential factor appearing in the first term of (1. 1) may have the value close to unity, and its value will decrease with increasing t_0. Let us choose t_0 such, that the first term in (1. 1) can be neglected for the accuracy required, i. e. let us limit ourselves to the effective time interval equal to $t - t_0$ and to the corresponding effective ray length of ξ_i given by the relation $f(t, x_i, \xi_i)$.

If function f^+ undergoes little change on the selected interval $t - t_0$ on which the exponential factor of the integrand function changes its value from unity to practically zero, then using the theoretical basis of the Laplace's method we can infer that the value of the integral is governed, essentially, by the behavior of f^+ near $\tau = t$. If we assume that the function f^+ is practically constant on the interval $t - t_0$, then the integral in (1. 1) will become computable and Eq. (1. 1) will become, with the accuracy of up to the neglected terms,

$$f(t, x_i, \xi_i) = f^+(t, x_i, \xi_i) \tag{1.2}$$

or, utilizing the definition of f^+ we can obtain

$$\int (f'f_1' - ff_1)\, g\, d\sigma\, d\xi_1 = 0 \tag{1.3}$$

Assuming that the above arguments hold for all velocities ξ_i, i. e. if for all ξ_i such effective time interval $t - t_0$ and such domain of spatial variables (domain of dependence of f) can be chosen that f^+ is practically constant within it, then (1. 3) will hold for all ξ_i and this will in turn imply that the molecular velocity distribution should be locally Maxwellian within this region

$$f = f^{(0)} = \frac{n}{(2\pi RT)^{3/2}} \exp\frac{-c^2}{2RT}, \qquad \begin{matrix} c^2 = c_1^2 + c_2^2 + c_3^2 \\ c_i = \xi_i - u_i \end{matrix} \tag{1.4}$$

Here n, T and u_i are the local density, temperature and the gas velocity, and they are no longer defined by the kinetic equation, but by the equations of conservation of the flow field.

In this manner we arrive at the purely hydrodynamic description of the motion of gas. Kinetic equations show that the motion of gas can be regarded as a motion of an aggregate of fluid particles. Each fluid particle (infinitessimally small from the macroscopic point of view, but of the linear dimension larger than the maximum free path of a molecule) is in a local state of thermodynamic equilibrium and the main contribution to this equilibrium state comes from the collisions within the fluid particle, in other words, the fluid particle in question can be regarded as a closed thermodynamic system. Its density, temperature and velocity are governed by its surroundings, i.e. by the complete field of flow, and should be obtained from the equations of conservation.

We can easily see that in this, or any other case, the motion of gas could be obtained by direct integration of the Boltzmann equation. However, the connection between the elementary volumes dx_i and the states of the gas at different instants of time, must be maintained during the process and this in turn demands that the dimensions of dx_i and dt employed in numerical integration should be much smaller than those obtained from the macroscopic considerations. Transition from the Boltzmann equation to the equations of conservation implies a change to a higher scale.

2. Let us now suppose that f^+ varies little within its region of functional dependence. Let us pass in the integral (1.1) to a new variable of integration

$$\alpha = \int_{\tau}^{t} N_\tau d\tau, \qquad \alpha_0 = \int_{t_0}^{t} N_\tau d\tau \tag{2.1}$$

Expanding f^+ near the point t, x_i into a Taylor series containing only the terms linear in α, inserting the resulting expression of f into (1.1) and integrating, we obtain, to within the required accuracy,

$$f(t, x_i, \xi_i) = f^+(t, x_i, \xi_i) - \frac{1}{N}\left(\frac{\partial f^+}{\partial t} + \xi_i \frac{\partial f^+}{\partial x_i}\right)\Bigg|_{t, x_i} \tag{2.2}$$

Let us find the correction t_0 to the principal term $f^{(0)}$ of the asymptotic expansion of f. Since we can assume that $f = f^+ = f^{(0)}$ when $\frac{1}{N}\left|\frac{\partial f^+}{\partial t} + \xi_i \frac{\partial f^+}{\partial x_i}\right| \ll f^+$ we can be justified in saying that the function f can be represented by

$$f = f^{(0)}(1 + \varphi) \tag{2.3}$$

where φ denotes a small relative deviation from the locally Maxwellian distribution.

Inserting (2.3) into (2.2) and utilizing the definition of f^+ we obtain, on linearizing,

$$\int (\varphi' + \varphi_1' - \varphi - \varphi_1) f^{(0)} f_1^{(0)} g d\sigma d\xi_1 = \frac{\partial f^{(0)}}{\partial t} + \xi_i \frac{\partial f^{(0)}}{\partial x_i} \tag{2.4}$$

In accordance with the required degree of accuracy we have replaced f^+ under the differentiation signs by $f^{(0)}$ The accuracy of (2.4) compares with that of the intermediate equation which appears, when the Enskog method is applied to the Boltzmann equation. Assuming further that, just as in the Enskog method, first five moments of the distribution function are given by the first term of (2.3), we can write $\partial f^{(0)} / \partial t$ in terms of the derivatives $\partial n / \partial t$, $\partial u_i / \partial t$ and $\partial T / \partial t$, and the latter in terms of the spatial derivatives of the same quantities. This follows from the equations of conservation containing only the moments corresponding to the zero'th approximation. Thus we obtain the following first approximation to the Enskog equation defining φ

$$\int (\varphi' + \varphi_1' - \varphi - \varphi_1) f^{(0)} f_1^{(0)} g d\sigma d\xi_1 = D^{(1)} \tag{2.5}$$

where $D^{(1)}$ is used in the sense of the well known monograph of Chapman and Cowling [1].

We note that in (2.4), individual derivatives corresponding to the liquid particle should have been used instead of the partial derivatives with respect to time, i.e. $\partial f^{(0)} / \partial t$ etc. Enskog's method exaggerates artificially the importance of the partial derivatives with respect to time. Indeed, in the case of steady state flows these partial derivatives are all equal to zero, and the results of the method can be successfully applied to all steady state problems, provided that relevant conditions are met. Individual derivatives are the only ones that are important and all arguments concerning the method of separation of the derivatives in the Enskog's method should be taken as referring to the individual derivatives. This is obvious from the physical point of view, since the Enskog method leads to the hydrodynamic representation of the motion of gas.

Of course, instead of confining ourselves to the first Enskog approximation for f, we could attempt to obtain a more accurate asymptotic representation by taking into account the second derivatives of f^* in the Taylor expansion. This would have yielded the Burnett approximation (it would not take us into the higher rarefaction region, since the effective domain of the functional dependence would not be altered). The question whether this method would yield better accuracy in the general case is open to doubt, since this would demand the smoothness of f^+ and consequently of f beyond that given by the Boltzmann equation.

BIBLIOGRAPHY

1. Chapman, S. and Cowling, T. G., Mathematical Theory of Nonuniform Cases. Cambridge University Press, 1952.
2. Kogan, M. N., Dynamics of a Rarefied Gas. M., "Nauka", 1967.
3. Struminskii, V. V., On a method of solution of the Boltzmann equation. Dokl. Akad. Nauk SSSR, Vol. 158, №2, 1964.
4. Evgrafov, M. A., Asymptotic Estimates and Entire Functions, 2nd ed., M., Fizmatgiz, 1962.

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APPLICATION OF THE LEAST SQUARES METHOD TO A MODEL WHICH IS NONLINEAR WITH RESPECT TO THE PARAMETERS

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We consider the application of the least squares method to models which are nonlinear with respect to the parameters and for which a linearizing transformation Ψ exists (such problems e. g. arise in the experimental determination of the parameters of the exponential criterial equations). We prove that the values of the parameters obtained after the transformation deviate from the required values, and we show the logarithmic transformation as an example illustrating the method used to obtain formulas yielding the estimates of these errors.

Iteration method which we propose, retains the advantages of the computations based on the linear model, but removes the error mentioned above, and we give the sufficient conditions for its convergence. To illustrate the method, we use empirical data on the mass transfer at the wall in a turbulent fluid flow at large Schmidt numbers.

Let us consider a random variable function of the form $Y(x) = f(x) + \xi(x)$ (without loss of generality, we can assume that this function is defined for $a \leqslant x \leqslant b$ and is equal to zero outside this interval), where $f(x) = \langle Y(x) \rangle$ and $\xi(x)$ is a stationary random function such, that $\langle \xi(x) \rangle = 0$. Usually, the problem of estimating the regression curve $f(x)$